# Quantum Lévy processes on locally compact quantum groups

Adam Skalski (joint work with Martin Lindsay)

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# Classical Lévy processes on semigroups

Let $G$ be a (semi)group. A *Lévy process on $G$* is a family of random variables $\{X_t : t \in \mathbb{R}_+\}$ with values in $G$ such that

(L1) $X_{r,t} = X_{r,s}X_{s,t}$, where $X_{s,t} = X_s^{-1}X_t$ denote the increments;

(L2) $X_{t,t} = e$ (almost surely);

(L3) the increments are identically distributed;

(L4) the increments are independent;

(L5) $X_t$ converges weakly to $\delta_e$ as $t \to 0^+$.

To define quantum Lévy processes we need to rephrase the definition above in terms of the action induced by random variables $X_t$ on some algebra of functions on $G$. Let $F(G)$ - all $\mathbb{C}$-valued functions on $G$.

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# Algebra of coefficients of unitary representations of a compact group

For a compact group $G$, let $\mathrm{Rep}(G)$ denote the algebra of coefficients of finite-dimensional (unitary) representations of $G$:

$$f \in \mathrm{Rep}(G) \iff \exists_{(\pi,\mathsf{H})-\text{ f.d. rep. of } G} \ \exists_{\xi,\eta\in\mathsf{H}} \quad f = \langle \xi, \pi(\cdot)\eta \rangle.$$

It is easy to check that the operation $\Delta : F(G) \to F(G \times G)$ defined by

$$\Delta(f)(g,h) = f(gh), \quad g,h \in G, f : G \to \mathbb{C},$$

satisfies

$$\Delta(\mathrm{Rep}(G)) \subset \mathrm{Rep}(G) \odot \mathrm{Rep}(G).$$

We also define a character $\epsilon : F(G) \to \mathbb{C}$ by

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# *-bialgebras

**Definition**

A (unital) *-bialgebra is a unital *-algebra $\mathcal{A}$ equipped with unital *-homomorphisms $\Delta : \mathcal{A} \to \mathcal{A} \odot \mathcal{A}$ (*coproduct*) and $\epsilon : \mathcal{A} \to \mathbb{C}$ (*counit*) such that

$$(\mathrm{id}_{\mathcal{A}} \odot \Delta) \circ \Delta = (\Delta \odot \mathrm{id}_{\mathcal{A}}) \circ \Delta,$$

$$(\mathrm{id} \odot \epsilon) \circ \Delta = (\epsilon \odot \mathrm{id}_{\mathcal{A}}) \circ \Delta = \mathrm{id}_{\mathcal{A}}.$$

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A *quantum probability space* $(B, \omega)$ is a unital ${}^*$-algebra with a state.

# Algebraic quantum Lévy processes [following Accardi, Schürmann and von Waldenfels]

A *quantum Lévy process* on a $^*$-bialgebra $\mathcal{A}$ over a *quantum probability space* $(\mathcal{B}, \omega)$ is a family $\left\{j_{s,t} : \mathcal{A} \to \mathcal{B} \middle|; 0 \le s \le t\right\}$ of unital $*$-homomorphisms such that

(Q1) $j_{r,t} = m \circ (j_{r,s} \odot j_{s,t}) \circ \Delta, \quad (0 \le r \le s \le t);$

(Q2) $j_{t,t}(a) = \epsilon(a)1_{\mathcal{B}};$

(Q3) $\omega \circ j_{s,t} = \omega \circ j_{0,t-s}$ for $0 \le s \le t$;

(Q4) $\{j_{s_i,t_i}(\mathcal{A}) : i = 1, \ldots n\}$ commute, and

$$\omega\left(\prod_{i=1}^{n} j_{s_i,t_i}(a_i)\right) = \prod_{i=1}^{n} \omega\left(j_{s_t,t_i}(a_i)\right)$$

whenever $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathcal{A}$ and the intervals $[s_1, t_1[, \ldots, [s_n, t_n[$ are disjoint;

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# Schürmann reconstruction theorem

In fact all 'stochastic' information about the quantum Lévy process is contained in its *Markov semigroup*:

$$\lambda_t := \omega \circ j_{0,t} \quad , t \in \mathbb{R}_+ .$$

Its generator

$$\gamma = \lim_{t \to 0^+} \frac{\lambda_t - \epsilon}{t}$$

is a hermitian, conditionally positive (positive on the kernel of the counit) functional on $\mathcal{A}$ vanishing at 1.

**Theorem (Schürmann)**

Each functional $\gamma$ as above is a generator of a Markov semigroup of a unique (up to stochastic equivalence) quantum Lévy process. This process can be concretely realised on a symmetric Fock space as a solution of a coalgebraic QSDE (quantum stochastic differential equation).

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# Compact quantum (semi)-groups

Instead of the algebra Rep($G$) we could consider all continuous functions on $G$. Then in general

$$\Delta(C(G)) \not\subseteq C(G) \odot C(G).$$

Denote the minimal tensor product of $C^*$-algebras by $\otimes$.

**Definition**

A unital $C^*$-bialgebra (in other words, a compact quantum semigroup) is a unital $C^*$-algebra A equipped with unital $*$-homomorphisms $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ (*coproduct*) and $\epsilon : \mathsf{A} \to \mathbb{C}$ (*counit*) such that

$$(\mathrm{id}_{\mathsf{A}} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_{\mathsf{A}}) \circ \Delta,$$

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# Complete boundedness

Warning: the multiplication $m$ almost never extends to a bounded map from $\mathsf{A} \otimes \mathsf{A}$ to $\mathsf{A}$!

In fact to define maps of the type $T \otimes 1_{\mathsf{A}} : \mathsf{A} \otimes \mathsf{A} \to \mathsf{B} \otimes \mathsf{A}$ we need to know that $T$ is *completely bounded*. As all bounded functionals in $\mathsf{A}^*$ are automatically completely bounded (and $\mathbb{C} \otimes \mathbb{C} \approx \mathbb{C}$), we can define for $\lambda, \mu \in \mathsf{A}^*$ their convolution:

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# Weak definition of quantum Lévy processes

A *weak quantum Lévy process* on a $C^*$-bialgebra A over a quantum probability space $(\mathcal{B}, \omega)$ is a family $\big(j_{s,t} \colon \mathsf{A} \to \mathcal{B}\big)_{0 \leq s \leq t}$ of unital *-homomorphisms such that if $\lambda_{s,t} := \omega \circ j_{s,t}$ then

(wQ1) $\lambda_{r,t} = \lambda_{r,s} \star \lambda_{s,t}$;

(wQ2) $\lambda_{t,t} = \epsilon$;

(wQ3) $\lambda_{s,t} = \lambda_{0,t-s}$;

(wQ4)
$$\omega\left(\prod_{i=1}^{n} j_{s_i,t_i}(a_i)\right) = \prod_{i=1}^{n} \lambda_{s_i,t_i}(a_i)$$
whenever $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathsf{A}$ and the intervals $[s_1, t_1[, \ldots, [s_n, t_n[$ are disjoint;

(wQ5) $\lambda_{0,t}(a) \to \epsilon(a)$ pointwise as $t \to 0$.

# Reconstruction theorem and Markov-regularity

We still have a version of the Schürmann reconstruction theorem, but only for *Markov regular* processes, i.e. those whose Markov semigroup $\{\lambda_{0,t} : t \in \mathbb{R}_+\}$ is norm continuous.

If A is a compact quantum group in the sense of Woronowicz, then it contains a canonical dense $^*$-bialgebra $\mathcal{A}$ – the algebra of coefficients of all unitary corepresentations of A. Thus to study quantum Lévy processes on compact quantum groups we can effectively use the algebraic theory. This is no longer true for locally compact quantum groups.

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Let $G$ be a locally compact group. This time the natural $C^*$-algebra of functions to be considered is $C_0(G)$ – the algebra of continuous functions vanishing at infinity. But...

$$\Delta(C_0(G)) \not\subseteq C_0(G) \otimes C_0(G) \approx C_0(G \times G)$$

In fact

$$\Delta(C_0(G)) \subset C_b(G \times G)$$

(with $C_b(G \times G)$ - the algebra of all continuous bounded functions on $G \times G$).

Note that

$$C_b(G \times G) = \{f : G \to \mathbb{C} : \forall_{f' \in C_0(G)}\, ff' \in C_0(G)\}.$$

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# Multiplier algebras

**Definition**

Let A be a $C^*$-algebra. The multiplier algebra of A, denoted $M(\mathrm{A})$, is the largest unital $C^*$-algebra in which A is an essential ideal (in other words, the largest reasonable unitisation of A).

If $\mathrm{A} \subset B(\mathrm{h})$, then

$$M(\mathrm{A}) \approx \{T \in B(\mathrm{h}) : \forall_{a \in \mathrm{A}}\ Ta \in \mathrm{A},\ aT \in \mathrm{A}\}$$

If $X$ is a locally compact space, then

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If A is unital, then $M(\mathrm{A}) = \mathrm{A}$.

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# Strict topology ...

Each multiplier algebra is equipped with *strict* topology, generated by the family of seminorms $\{r_a, l_a : a \in \mathrm{A}\}$, where

$$r_a(m) = \|ma\|, \quad l_a(m) = \|am\|, \quad m \in M(\mathrm{A}).$$

Note that the strict topology of $M(\mathrm{A})$ depends on A. But...

**Theorem (Woronowicz)**

If A is separable, then $\mathrm{A} = \{m \in M(\mathrm{A}) : mM(\mathrm{A}) \text{ is separable}\}$; hence in the separable case $M(\mathrm{A})$ determines A.

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# . . . and strict extensions

**Definition**

A linear map $T : \mathsf{A}_1 \to M(\mathsf{A}_2)$ is called strict if it is bounded and strictly continuous on bounded sets.

Each strict map $T : \mathsf{A}_1 \to M(\mathsf{A}_2)$ admits a unique strict extension to a map $\widetilde{T} : M(\mathsf{A}_1) \to M(\mathsf{A}_2)$, with $\|\widetilde{T}\| = \|T\|$.

A $^*$-homomorphism $T : \mathsf{A}_1 \to M(\mathsf{A}_2)$ is called *nondegenerate* if $T(\mathsf{A}_1)\mathsf{A}_2$ is dense in $\mathsf{A}_2$; it is then strict. A completely positive map is strict if and only if for some approximate unit $(e_i)_{i \in I}$ the net $(T(e_i))_{i \in I}$ is strictly convergent.

**Theorem**

Every completely bounded map $T : \mathsf{A} \to B(\mathsf{h}) \approx M(K(\mathsf{h}))$ is strict.

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# Locally compact quantum (semi)-groups

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A multiplier $C^*$-bialgebra (in other words, a locally compact quantum semigroup) is a $C^*$-algebra $\mathsf{A}$ equipped with nondegenerate $^*$-homomorphisms $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ (*coproduct*) and $\epsilon : \mathsf{A} \to \mathbb{C}$ (*counit*) such that

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Examples: $C_0(G)$ for a locally compact (semi)group $G$, locally compact quantum groups of Kustermans and Vaes (in the universal setting).

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# Weak definition of quantum Lévy processes

A *weak quantum Lévy process* on a multiplier $C^*$-bialgebra A over a $C^*$-algebra with a state $(\mathsf{B}, \omega)$ is a family $\big(j_{s,t} \colon \mathsf{A} \to M(\mathsf{B})\big)_{0 \le s \le t}$ of nondegenerate *-homomorphisms such that if $\lambda_{s,t} := \omega \circ j_{s,t}$ then

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whenever $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathsf{A}$ and the intervals $[s_1, t_1[, \dots, [s_n, t_n[$ are disjoint;

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# Convolution semigroups of states and Markov regularity

A family of states $(\lambda_t)_{t\geq 0}$ on A is called a convolution semigroup of states if

$$\lambda_{t+s} = \lambda_t \star \lambda_s, \quad t, s \geq 0$$

and it is weakly continuous at 0:

$$\forall_{a\in \mathsf{A}} \quad \lambda_t(a) \stackrel{t\to 0^+}{\longrightarrow} \lambda_0(a) = \epsilon(a).$$

It is called norm continuous if

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then it has a *generator* $\gamma \in \mathsf{A}^*$ - a hermitian, conditionally positive functional such that $\tilde{\gamma}(1) = 0$ and

$$\lambda_t = \sum_{k=0}^{\infty} \frac{(t\gamma)^{*k}}{k!}, t > 0.$$

A quantum Lévy process on A is called Markov regular if its *Markov semigroup* $(\omega \circ j_{0,t})_{t\geq 0}$ is norm continuous.

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A quantum Lévy process on A is called Markov regular if its *Markov semigroup* $(\omega \circ j_{0,t})_{t\geq 0}$ is norm continuous.

# Convolution semigroups of states and Markov regularity

A family of states $(\lambda_t)_{t\geq 0}$ on A is called a convolution semigroup of states if

$$\lambda_{t+s} = \lambda_t \star \lambda_s, \quad t, s \geq 0$$

and it is weakly continuous at 0:

$$\forall_{a\in \mathsf{A}} \quad \lambda_t(a) \xrightarrow{t\to 0^+} \lambda_0(a) = \epsilon(a).$$

It is called norm continuous if

$$\lambda_t \xrightarrow{t\to 0^+} \epsilon \qquad \text{in norm;}$$

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# Reconstruction theorem

The following theorem is a version of the Schürmann's reconstruction theorem for the topological context of quantum Lévy processes on locally compact quantum semigroups.

**Theorem**

Let $\gamma \in \mathsf{A}^*$ be real, conditionally positive and satisfy $\widetilde{\gamma}(1) = 0$. Then there is a nondegenerate representation $(\pi, \mathsf{h})$ of $\mathsf{A}$ and vector $\eta \in \mathsf{h}$ such that $\gamma = \langle \eta, \pi(\cdot)\eta \rangle$.
Moreover there is a (Markov-regular) Fock space quantum Lévy process with generating functional $\gamma$.

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# Semigroups on the level of the algebra

Let us consider again a weakly continuous convolution semigroup of functionals $(\lambda_t)_{t\geq 0}$ on a multiplier $C^*$-bialgebra A. To study general non-Markov regular quantum Lévy processes we need to understand better unbounded *generators* of such semigroups.

**Definition**

Functional $\gamma : \operatorname{Dom}\gamma \subset \mathsf{A} \to \mathbb{C}$ defined by

$$\operatorname{Dom}\gamma := \left\{a \in \mathsf{A} : \lim_{t\to 0^+} \frac{\lambda_t(a) - \epsilon(a)}{t} \text{ exists}\right\};$$
$$\gamma(a) := \lim_{t\to 0^+} \frac{\lambda_t(a) - \epsilon(a)}{t}, \quad a \in \operatorname{Dom}\gamma,$$

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# Residual vanishing at infinity and its consequences

## Definition

A multiplier $C^*$-bialgebra A has the 'residual vanishing at infinity' property if

$$(\mathsf{A} \otimes 1_{M(\mathsf{A})})\Delta(\mathsf{A}) \subset \mathsf{A} \otimes \mathsf{A} \quad \text{and} \quad (1_{M(\mathsf{A})} \otimes \mathsf{A})\Delta(\mathsf{A}) \subset \mathsf{A} \otimes \mathsf{A}.$$

Example: unital $C^*$-bialgebras, locally compact quantum groups in the sense of Kustermans and Vaes.

### Theorem

If $(\lambda_t)_{t\geq 0}$ is a convolution semigroup of functionals on a (multiplier) $C^*$-bialgebra A with the 'residual vanishing at infinity' property, then

- the generating functional $\gamma$ is densely defined;
- $\gamma$ determines $(\lambda_t)_{t\geq 0}$;
- $(\lambda_t)_{t\geq 0}$ is continuous if and only if $\gamma$ is bounded (in which case $\operatorname{Dom}\gamma = \mathrm{A}$).

### Problem

Find Hille-Yoshida type conditions characterising generating functionals.

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# Discrete quantum groups

**Definition**

A multiplier $C^*$-bialgebra A is called a discrete quantum semigroup if A is a direct sum of matrix algebras: $A \approx \bigoplus_{i \in I} M_{n_i}$.

Example: discrete quantum groups considered by Kustermans, Vaes and Van Daele.

**Theorem**

Every convolution semigroup of states on a discrete quantum semigroup is automatically norm continuous.

**Corollary**

Every quantum Lévy process on a discrete quantum semigroup is of compound Poisson type.

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# Bibliography

**Classical Lévy processes**

D. Applebaum, "Lévy processes and stochastic calculus ", 2009

**Quantum Lévy processes in the algebraic framework**

M. Schürmann, "White noise on bialgebras", 1993.

U. Franz, Lévy processes on quantum groups and dual groups, 2005.

**Quantum Lévy processes in the topological framework**

M. Lindsay and A.S., Quantum stochastic convolution cocycles I-III, 2005-2010.

**Generators of convolution semigroups**

M. Lindsay and A.S., Convolution semigroups of states, 2010